A Structural Closed-Form Representation of the Odd Collatz Iteration

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Abstract

We derive an exact closed-form expression for the k-step iteration of the odd Collatz map

$$T(x) = \frac{3x+1}{2^{v_2(3x+1)}},$$

and obtain a structural cycle condition

$$(2^A - 3^k)x_0 = S,$$

where A and S encode, respectively, the cumulative 2-adic reductions and the weighted contribution of all "+1" operations. This formulation separates the multiplicative and additive components of the Collatz dynamics and provides a representation independent of the binary positional system.

1 Introduction

Consider the odd Collatz map

$$x_{i+1} = \frac{3x_i + 1}{2^{a_i}}, \qquad a_i = v_2(3x_i + 1),$$

where all x_i remain odd. We derive an exact expression for x_k in terms of x_0, the exponents a_i, and a weighted sum of all additive increments. This yields a structural algebraic condition for cycles.

2 Exact k-step formula

After k iterations of the odd Collatz map,

$$x_k = \frac{3^k x_0 + S}{2^A},$$

where

$$A = \sum_{i=0}^{k-1} a_i,$$

and

$$S = \sum_{j=0}^{k-1} 3^{k-1-j} \cdot 2^{\sum_{i=0}^{j-1} a_i}.$$

Proof. Base case $k = 1$:

$$x_1 = \frac{3x_0 + 1}{2^{a_0}},$$

so $A = a_0$ and $S = 1$.
Inductive step: assume

$$x_k = \frac{3^k x_0 + S_k}{2^{A_k}}.$$

Then

$$x_{k+1} = \frac{3x_k + 1}{2^{a_k}} = \frac{3(3^k x_0 + S_k)/2^{A_k} + 1}{2^{a_k}} = \frac{3^{k+1} x_0 + 3S_k + 2^{A_k}}{2^{A_k + a_k}}.$$

Thus

$$A_{k+1} = A_k + a_k, \qquad S_{k+1} = 3S_k + 2^{A_k}.$$

Unrolling the recurrence yields the stated closed form for S. □

3 Cycle condition

A cycle of length k satisfies $x_k = x_0$. Substituting the closed form:

$$x_0 = \frac{3^k x_0 + S}{2^A},$$

so

$$(2^A - 3^k) x_0 = S.$$

For normalized $x_0 = 1$:

$$3^k + S = 2^A.$$

4 Example: $k = 3$

We illustrate the closed-form formula using the initial value $x_0 = 7$.

Step-by-step iteration

$$\begin{aligned} 3 \cdot 7 + 1 = 22 = 2 \cdot 11, &\qquad a_0 = 1, &\qquad x_1 = 11, \\ 3 \cdot 11 + 1 = 34 = 2 \cdot 17, &\qquad a_1 = 1, &\qquad x_2 = 17, \\ 3 \cdot 17 + 1 = 52 = 4 \cdot 13, &\qquad a_2 = 2, &\qquad x_3 = 13. \end{aligned}$$

Thus,

$$A = a_0 + a_1 + a_2 = 1 + 1 + 2 = 4.$$

Computing the structural sum S

$$S = 3^2 2^0 + 3^1 2^1 + 3^0 2^2 = 9 + 6 + 4 = 19.$$

Closed-form verification

$$x_3 = \frac{3^3 \cdot 7 + 19}{2^4} = \frac{189 + 19}{16} = \frac{208}{16} = 13.$$

This matches the iterative computation exactly.

5 Example: impossibility of a 2-cycle

Assume a 2-cycle with $x_0 = 1$. The cycle condition gives:

$$3^2 + S = 2^A.$$

The left-hand side is odd because S is a sum of terms of the form

$$3^{k-1-j} 2^{\sum a_i},$$

each of which is odd.
The right-hand side 2^A is even.
Thus no equality is possible, and no 2-cycle exists.

6 Example: $k = 4$ with mixed exponents

We illustrate the closed-form formula on a longer trajectory with mixed 2-adic exponents, using $x_0 = 5$.

Step-by-step iteration

$$\begin{aligned}
3 \cdot 5 + 1 = 16 = 2^4 \cdot 1, &\qquad a_0 = 4, &\qquad x_1 = 1,\\
3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, &\qquad a_1 = 2, &\qquad x_2 = 1,\\
3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, &\qquad a_2 = 2, &\qquad x_3 = 1,\\
3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, &\qquad a_3 = 2, &\qquad x_4 = 1.
\end{aligned}$$

Thus the sequence is:

$$x_0 = 5, \quad x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 1.$$

Cumulative exponent A

$$A = a_0 + a_1 + a_2 + a_3 = 4 + 2 + 2 + 2 = 10.$$

Structural sum S

We compute each term of

$$S = \sum_{j=0}^{3} 3^{3-j} \cdot 2^{\sum_{i=0}^{j-1} a_i}.$$

$$\begin{aligned}
j = 0: &\quad 3^3 2^0 = 27,\\
j = 1: &\quad 3^2 2^{a_0} = 9 \cdot 2^4 = 144,\\
j = 2: &\quad 3^1 2^{a_0 + a_1} = 3 \cdot 2^6 = 192,\\
j = 3: &\quad 3^0 2^{a_0 + a_1 + a_2} = 1 \cdot 2^8 = 256.
\end{aligned}$$

Thus,

$$S = 27 + 144 + 192 + 256 = 619.$$

Closed-form verification

The closed-form formula predicts:

$$x_4 = \frac{3^4 \cdot 5 + 619}{2^{10}} = \frac{405 + 619}{1024} = \frac{1024}{1024} = 1.$$

This matches the iterative computation exactly.

7 Related Work

Several authors have studied linear forms and structural aspects of the Collatz map, but, to the best of our knowledge, no prior work provides the exact closed-form decomposition of both the cumulative 2-adic exponent A and the additive term S as given here, nor the resulting structural cycle equation

$$(2^A - 3^k)x_0 = S.$$

8 Conclusion

We derived an exact algebraic representation of the odd Collatz iteration and a structural cycle condition. This decomposition separates multiplicative, additive, and 2-adic components of the dynamics and appears not to be present in existing literature.

References

[1] J. C. Lagarias, *The* $3x + 1$ *problem and its generalizations*, American Mathematical Monthly, 92(1), 1985.